

## ON THE COX RING OF BLOWING UP THE DIAGONAL

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ABSTRACT. We compute the Cox rings of the blow-ups  $\mathrm{Bl}_\Delta(X' \times X')$  and  $\mathrm{Bl}_\Delta(\mathbb{P}_1^n)$  where  $X'$  is a product of projective spaces and  $\Delta$  is the (generalised) diagonal.

## 1. INTRODUCTION

To a complete, normal variety with finitely generated divisor class group one can associate its Cox ring. In recent literature it has been discussed how the Cox ring behaves under blow-ups. In particular, it is of interest whether finite generation is preserved in this process, and, if so what a presentation in terms of generators and relations looks like, see for example [4, 5, 7, 8, 9, 10].

In the present note we employ the techniques developed in [2, 8] to compute the Cox rings of the following blow-ups. Let  $X' := \mathbb{P}_{n_1} \times \dots \times \mathbb{P}_{n_r}$  be a product of projective spaces and denote by  $\Delta_X \subseteq X := X' \times X'$  the diagonal. The variety  $X$  is spherical and  $\mathrm{Bl}_{\Delta_X}(X)$  inherits this property. Hence, it is known that the Cox ring  $\mathcal{R}(\mathrm{Bl}_{\Delta_X}(X))$  is finitely generated, see [1, 3]. Our first result is an explicit presentation.

**Theorem 1.1.** *The Cox ring  $\mathcal{R}(\mathrm{Bl}_{\Delta_X}(X))$  of the blow-up  $\mathrm{Bl}_{\Delta_X}(X)$  is isomorphic to the  $\mathbb{Z}^r \times \mathbb{Z}^r \times \mathbb{Z}$ -graded factor algebra  $R_X/I_X$  where*

$$\begin{aligned} R_X &:= \mathbb{K}[T_\infty, {}_rT_{ij}; \quad r = 1, \dots, r, \quad 0 \leq i < j \leq n_r + 2, \quad i \leq n_r], \\ I_X &:= I(1) + \dots + I(r), \end{aligned}$$

for every  $r = 1, \dots, r$  the ideal  $I(r)$  is generated by the twisted Plücker relations

$$\begin{aligned} {}_rT_{ij} T_\infty - {}_rT_{ik} {}_rT_{jk} + {}_rT_{il} {}_rT_{jk}; \quad 0 \leq i < j \leq n_r, \quad k = n_r + 1, \quad l = n_r + 2, \\ {}_rT_{ij} {}_rT_{kl} - {}_rT_{ik} {}_rT_{jl} + {}_rT_{il} {}_rT_{jk}; \quad 0 \leq i < j < k < l \leq n + 2, \quad k \leq n_r, \end{aligned}$$

and the grading of  $R_X/I_X$  is given by

$$\deg(T_\infty) = (0, 0, 1), \quad \deg({}_rT_{ij}) = \begin{cases} (e_r, 0, 0) & \text{if } j = n_r + 1, \\ (0, e_r, 0) & \text{if } j = n_r + 2, \\ (e_r, e_r, -1) & \text{else.} \end{cases}$$

In particular, the spectrum of the Cox ring  $\mathcal{R}(\mathrm{Bl}_{\Delta_X}(X))$  is the intersection of a product of affine Grassmannian varieties (w.r.t. the Plücker embedding) with a linear subspace.

As a second class of examples we treat the (non-spherical) blow-up of the variety  $Y := \mathbb{P}_1^n$  in the generalised diagonal  $\Delta_Y := \{(x, \dots, x); x \in \mathbb{P}_1\} \subseteq Y$ . Again we prove that the Cox ring of  $\mathrm{Bl}_{\Delta_Y}(Y)$  is finitely generated and we give an explicit presentation.

**Theorem 1.2.** *The Cox ring  $\mathcal{R}(\mathrm{Bl}_{\Delta_Y}(Y))$  of the blow-up  $\mathrm{Bl}_{\Delta_Y}(Y)$  is isomorphic to the  $\mathbb{Z}^{n+1}$ -graded factor algebra  $R_Y/I_Y$  where*

$$\begin{aligned} R_Y &:= \mathbb{K}[S_{ij}; 1 \leq i < j \leq n+2] \\ I_Y &:= \langle S_{ij}S_{kl} - S_{ik}S_{jl} + S_{il}S_{jk}; 1 \leq i < j < k < l \leq n+2 \rangle, \end{aligned}$$

and the grading of  $R_Y/I_Y$  is given by

$$\deg(S_{ij}) = \begin{cases} e_i & \text{if } i \leq n, j = n+1, n+2, \\ e_{n+1} & \text{if } i = n+1, j = n+2, \\ e_i + e_j - e_{n+1} & \text{else.} \end{cases}$$

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## 2. PROOFS OF THEOREMS 1.1 AND 1.2

Let us recall some definitions, for details see [1]. Let  $Z$  be a normal variety with free and finitely generated divisor class group  $K := \mathrm{Cl}(Z)$  and only constant invertible regular functions. Then we define its *Cox ring* as the  $K$ -graded  $\mathbb{K}$ -algebra

$$\mathcal{R}(Z) := \bigoplus_K \Gamma(Z, \mathcal{O}(D)).$$

If the Cox ring  $\mathcal{R}(Z)$  is finitely generated, we call its spectrum  $\overline{Z} := \mathrm{Spec}(\mathcal{R}(Z))$  the *total coordinate space* of  $Z$ . The  $K$ -grading of  $\mathcal{R}(Z)$  gives rise to an action of the quasitorus  $H_Z := \mathrm{Spec}(\mathbb{K}[K])$  on  $\overline{Z}$ . Moreover, there exists an open invariant subset, the *characteristic space*,  $\hat{Z} \subseteq \overline{Z}$  admitting a good quotient  $p_Z: \hat{Z} \rightarrow \hat{Z} // H_Z \cong Z$  for this action.

Before we enter the proofs we will sketch the methods developed in [2, 8]. Let  $Z$  be a toric variety with Cox construction  $p_Z: \hat{Z} \rightarrow Z$ , total coordinate space  $\overline{Z} = \mathbb{K}^r$  and an ample class  $w \in K$  in the divisor class group.

Now let  $\mathfrak{A}$  be a subscheme of  $Z$ ; we ask for the Cox ring of the blow-up  $\mathrm{Bl}_{\mathfrak{A}}(Z)$  of  $Z$  in  $\mathfrak{A}$ . By Cox' construction [6] the subscheme  $\mathfrak{A}$  arises from a homogeneous ideal  $\mathfrak{a} = \langle f_1, \dots, f_l \rangle$  in the  $K$ -graded Cox ring  $\mathcal{R}(Z)$ . For this consider the associated  $K$ -graded sheaf  $\tilde{\mathfrak{a}}$  on  $\overline{Z}$ ; then  $\mathfrak{A}$  is given by  $(p_{Z*}\tilde{\mathfrak{a}})_0$ . As a first step we embed  $Z$  into a larger toric variety  $Z_1$  such that the blow-up can be dealt with using methods from toric geometry. For this we consider the closed embedding

$$\overline{\pi}: \mathbb{K}^r \rightarrow \mathbb{K}^{r_1}; \quad z \mapsto (z, f_1(z), \dots, f_l(z)),$$

where  $r_1 := r + l$ . We endow  $\mathbb{K}[T_1, \dots, T_{r_1}]$  with a grading of  $K_1 := K$  by assigning to  $T_1, \dots, T_r$  the original  $K$ -degrees and setting  $\deg(T_{r+i}) := \deg f_i$  for the remaining variables. Then the quasitorus  $H_{Z_1} := \mathrm{Spec}(\mathbb{K}[K_1])$  acts on the affine space  $\overline{Z}_1 := \mathbb{K}^{r_1}$  and this makes  $\overline{\pi}$  equivariant. The class  $w \in K_1$  gives rise to an open subset  $\hat{Z}_1 \subseteq \overline{Z}_1$  and a toric variety  $Z_1 := \hat{Z}_1 // H_{Z_1}$ .

The closed embedding  $\overline{\pi}$  restricts to a closed embedding  $\hat{\pi}: \hat{Z} \rightarrow \hat{Z}_1$  of the corresponding characteristic spaces and then descends to a closed embedding  $\pi: Z \rightarrow Z_1$  of the respective quotients. The setting fits into the following commutative diagram.

$$\begin{array}{ccc}
\overline{Z} & \xrightarrow{\overline{\pi}} & \overline{Z}_1 \\
\uparrow & & \uparrow \\
\hat{Z} & \xrightarrow{\hat{\pi}} & \hat{Z}_1 \\
\downarrow \parallel H_Z & & \downarrow \parallel H_{Z_1} \\
Z & \xrightarrow{\pi} & Z_1
\end{array}$$

The idea is to compute the Cox ring of the proper transform  $Z'$  of  $Z \subseteq Z_1$  with respect to a toric blow-up of  $Z_1$ . The following lemma relates  $Z'$  to the blow-up  $\text{Bl}_{\mathfrak{A}}(Z)$ . Although the result was to be expected, we do not know of a reference and provide a proof.

**Lemma 2.1.** *Let  $\mathfrak{b} \subseteq \mathcal{O}(\overline{Z}_1)$  be a  $K_1$ -homogeneous ideal and let  $\mathfrak{B}$  be the corresponding subscheme of  $Z_1$ . Then the proper transform of  $Z \subseteq Z_1$  under the blow-up  $\text{Bl}_{\mathfrak{B}}(Z_1) \rightarrow Z_1$  is isomorphic to the blow-up of  $Z$  in the subscheme of  $Z$  associated to the  $K$ -homogeneous ideal  $\overline{\pi}^* \mathfrak{b} \subseteq \mathcal{O}(\overline{Z})$ .*

**Remark 2.2.** If we apply Lemma 2.1 in the case  $\mathfrak{b} := \langle T_{r+1}, \dots, T_{r_1} \rangle$ , then we obtain  $\mathfrak{a} = \overline{\pi}^* \mathfrak{b}$  and the proper transform  $Z'$  of  $Z \subseteq Z_1$  is the blow-up of  $Z$  in  $\mathfrak{A}$ . Moreover, if  $\mathfrak{a}$  is prime, then the associated subscheme  $\mathfrak{A}$  is the subvariety  $p_Z(V(\mathfrak{a}))$  and  $Z'$  is the ordinary blow-up of  $Z$  in  $p_Z(V(\mathfrak{a}))$ .

*Proof of Lemma 2.1.* First blow-ups are determined locally. We consider a suitable partial open cover of  $\hat{Z}_1$  and of the characteristic space  $\hat{Z}$ . Let  $w \in K = K_1$  be an ample class of  $Z$  as above. We set

$$\Gamma := \{ \gamma \in \{0, 1\}^{r_1}; \quad w \in \text{relint}(\text{cone}(\deg T_i; \text{ where } \gamma_i = 1)) \}.$$

Then  $\hat{Z}_1$  is covered by the  $H_{Z_1}$ -invariant sets  $\overline{Z}_{1\gamma} := \overline{Z}_1 \setminus V(T^\gamma)$  where  $\gamma \in \Gamma$ . We now determine a partial cover which already contains  $\overline{\pi}(\hat{Z})$ . For this we consider the subset

$$\Gamma' := \Gamma \cap (\{0, 1\}^r \times \{0\}^l) \subseteq \Gamma.$$

Then the corresponding open subvarieties cover the image of  $\overline{\pi}$ . More precisely, if we set  $\overline{Z}_\gamma := \overline{\pi}^{-1}(\overline{Z}_{1\gamma})$ , then we have

$$\hat{Z} = \bigcup_{\gamma \in \Gamma'} \overline{Z}_\gamma \quad \text{and hence} \quad \overline{\pi}(\hat{Z}) \subseteq \bigcup_{\gamma \in \Gamma'} \overline{Z}_{1\gamma}.$$

Moreover, we denote by  $Z_\gamma := \overline{Z}_\gamma // H_Z$  and  $Z_{1\gamma} := \overline{Z}_{1\gamma} // H_{Z_1}$  the respective quotient spaces and fix some  $\gamma \in \Gamma'$ . If we set  $I_1 \subseteq \mathbb{K}[T_1, \dots, T_{r_1}]$  as the ideal generated by all the  $T_{r+i} - f_i$ , then the image of  $\overline{\pi}$  is given by  $V(I_1)$ . The morphism  $\overline{\pi}$  factors into an isomorphism  $\overline{\pi}': \overline{Z} \rightarrow V(I_1)$  and a closed embedding  $\iota: V(I_1) \rightarrow \overline{Z}_1$ .

On the algebraic side we set  $A := \mathcal{O}(\overline{Z})$  and  $B := \mathcal{O}(\overline{Z}_1)$ . We write  $B_\gamma$ ,  $(B/I)_\gamma$  and  $A_\gamma$  for the localised algebras and  $B_{(0)}$ ,  $(B/I)_{(0)}$  and  $A_{(0)}$  for their respective homogeneous components of degree zero. Then the situation fits into the following

commutative diagrams.

$$\begin{array}{ccccc}
\bar{Z} & \xrightarrow{\bar{\pi}'} & V(I_1) & \xrightarrow{\bar{\iota}} & \bar{Z}_1 \\
\uparrow & & \uparrow & & \uparrow \\
\bar{Z}_\gamma & \xrightarrow{\pi'_\gamma} & V(I_1) \cap \bar{Z}_{1\gamma} & \xrightarrow{\iota_\gamma} & \bar{Z}_{1\gamma} \\
\downarrow & & \downarrow & & \downarrow \\
Z_\gamma & \xrightarrow{\pi'} & Z_\gamma & \xrightarrow{\iota} & Z_{1\gamma}
\end{array}
\quad
\begin{array}{ccccc}
A & \xleftarrow{\bar{\pi}'^*} & B/I_1 & \xleftarrow{\bar{\iota}^*} & B \\
\downarrow & & \downarrow & & \downarrow \\
A_\gamma & \xleftarrow{\pi'^*_\gamma} & (B/I_1)_\gamma & \xleftarrow{\iota^*_\gamma} & B_\gamma \\
\uparrow & & \uparrow & & \uparrow \\
A_{(0)} & \xleftarrow{\pi'^*} & (B/I_1)_{(0)} & \xleftarrow{\iota^*} & B_{(0)}
\end{array}$$

The proper transform of  $Z_\gamma \subseteq Z_{1\gamma}$  is the blow-up of  $Z_\gamma$  with center given by the affine scheme associated to the ideal  $\iota^* \mathfrak{b}_{(0)} \subseteq (B/I_1)_{(0)}$ . Our assertion then follows from the fact that in  $A_{(0)}$  the ideals  $\pi'^*(\iota^* \mathfrak{b}_{(0)})$  and  $(\bar{\pi}'^* \mathfrak{b})_{(0)}$  coincide.  $\square$

**Construction 2.3.** Let  $\mathfrak{b}$  be the ideal  $\langle T_{r+1}, \dots, T_{r_1} \rangle$  and  $Z' \rightarrow Z$  the proper transform of  $Z \subseteq Z_1$  with respect to the toric blow-up  $\text{Bl}_{\mathfrak{B}}(Z_1) \rightarrow Z_1$ . We turn to the problem of determining the Cox ring  $\mathcal{R}(Z')$ . For this we set  $r_2 := r_1 + 1$  and consider the  $r_1 \times r_2$ -matrix

$$A := [E_{r_1}, \mathbf{1}_l], \quad \text{where } \mathbf{1}_l := (\underbrace{0, \dots, 0}_r, \underbrace{1, \dots, 1}_l)^t.$$

The dual map  $A^*: \mathbb{Z}^{r_1} \rightarrow \mathbb{Z}^{r_2}$  yields a homomorphism  $\alpha^*$  of group algebras and a morphism  $\alpha: (\mathbb{K}^*)^{r_2} \rightarrow (\mathbb{K}^*)^{r_1}$ . Together with the canonical embeddings  $\iota_1^*$  and  $\iota_2^*$  we now have transfer the ideal

$$I_1 := \langle T_{r+i} - f_i; i = 1, \dots, l \rangle \subseteq \mathbb{K}[T_1, \dots, T_{r_1}]$$

by taking extensions and contractions via the construction

$$\begin{array}{ccccccc}
I_1 & & I'_1 := \langle \iota_1^* I_1 \rangle & & I'_2 := \langle \alpha^* I'_1 \rangle & & I_2 := \iota_2^{*-1} I'_2 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{K}[T_1, \dots, T_{r_1}] & \xrightarrow{\iota_1^*} & \mathbb{K}[T_1^{\pm 1}, \dots, T_{r_1}^{\pm 1}] & \xrightarrow{\alpha^*} & \mathbb{K}[T_1^{\pm 1}, \dots, T_{r_2}^{\pm 1}] & \xleftarrow{\iota_2^*} & \mathbb{K}[T_1, \dots, T_{r_2}]
\end{array}$$

and call the resulting ideal  $I_2$ . If we endow  $\mathbb{K}[T_1, \dots, T_{r_2}]$  with the grading of  $K_2 := K_1 \times \mathbb{Z}$  given by

$$\deg(T_i) := \begin{cases} (\deg_{K_1}(T_i), 0) & \text{for } 1 \leq i \leq r, \\ (\deg_{K_1}(T_i), -1) & \text{for } r+1 \leq i \leq r_1, \\ (0, 1) & \text{for } i = r_2, \end{cases}$$

then  $I_2$  is  $K_2$ -homogeneous and the following Proposition provides us with a criterion to show that  $I_2$  defines the desired Cox ring.

**Proposition 2.4** ([2, Proposition 3.3, Corollary 3.4]). *If in the  $K_2$ -graded ring  $R_2 := \mathbb{K}[T_1, \dots, T_{r_2}]/I_2$  the variable  $T_{r_2}$  is prime and does not divide a  $T_i$  with  $1 \leq i \leq r_1$ , then  $R_2$  is Cox ring of the proper transform  $Z'$ .*

We return to our two cases of  $X = X' \times X'$  and  $Y = \mathbb{P}_1^n$ . Both of them are toric varieties, their respective Cox rings are polynomial rings and the total coordinate spaces are

$$\bar{X} = \bigoplus_{r=1}^r (\mathbb{K}^{n_r+1} \oplus \mathbb{K}^{n_r+1}) \quad \text{and} \quad \bar{Y} = \underbrace{\mathbb{K}^2 \oplus \dots \oplus \mathbb{K}^2}_n.$$

On  $\bar{X}$  we will label the coordinates of the  $r$ -th factor with  ${}_r T_{ij}$  where  $i = 0, \dots, n_r$  and  $j = n_r + 1, n_r + 2$ . On  $\bar{Y}$  we will use the notation  $S_{ij}$  for the coordinates where similarly  $i = 1, \dots, n$  and  $j = n + 1, n + 2$ .

The first step is to determine generators for the vanishing ideals of the generalised diagonals  $\Delta_X$  and  $\Delta_Y$  in the respective Cox rings, i.e. the ideals

$$\mathfrak{a}_X := I(p_X^{-1}(\Delta_X)) \subseteq \mathcal{O}(\overline{X}) \quad \text{and} \quad \mathfrak{a}_Y := I(p_Y^{-1}(\Delta_Y)) \subseteq \mathcal{O}(\overline{Y}).$$

**Lemma 2.5.** *As above let  $\mathfrak{a}_X$  and  $\mathfrak{a}_Y$  be the ideals of the generalised diagonals  $\Delta_X$  and  $\Delta_Y$  in the respective Cox rings. Both of them are prime and they are generated by the following elements.*

(i) *The ideal  $\mathfrak{a}_X$  is generated by the  $2 \times 2$ -minors of the matrices*

$$\begin{bmatrix} {}_rT_{0,n_r+1} & {}_rT_{1,n_r+1} & \cdots & {}_rT_{n_r,n_r+1} \\ {}_rT_{0,n_r+2} & {}_rT_{1,n_r+2} & \cdots & {}_rT_{n_r,n_r+2} \end{bmatrix}, \quad r = 1, \dots, \mathbf{r}.$$

(ii) *The ideal  $\mathfrak{a}_Y$  is generated by the  $2 \times 2$ -minors of the matrix*

$$\begin{bmatrix} S_{1,n+1} & S_{2,n+1} & \cdots & S_{n,n+1} \\ S_{1,n+2} & S_{2,n+2} & \cdots & S_{n,n+2} \end{bmatrix}.$$

The idea of the proof is to execute the computations on the respective tori. For future reference let us make the following remark.

**Remark 2.6.** Let  $\iota: (\mathbb{K}^*)^n \rightarrow \mathbb{K}^n$  be the canonical open embedding and  $\iota^*$  its comorphism. If  $I \subseteq \mathbb{K}[T_1, \dots, T_n]$  a prime ideal not containing any of the variables  $T_i$ , then  $(\iota^*)^{-1}(\iota^*(I)) = I$  holds.

*Proof of Lemma 2.5.* Let  $p_Z: \hat{Z} \rightarrow Z$  be the Cox construction of a toric variety  $Z$  and  $\overline{Z}$  its total coordinate space. We view the toric morphism  $p_Z$  as a morphism  $T_{\hat{Z}} \rightarrow T_Z$  of the openly embedded dense tori. Moreover, we denote by  $\Delta \subseteq Z$  a subvariety with  $\Delta = \overline{\Delta \cap T_Z}$  and write  $\iota': \hat{Z} \rightarrow \overline{Z}$  and  $\iota: T_{\hat{Z}} \rightarrow \overline{Z}$  for the canonical open embeddings.

$$\begin{array}{ccccc} & & \iota & & \\ & \curvearrowright & & \curvearrowright & \\ T_{\hat{Z}} & \xrightarrow{\quad} & \hat{Z} & \xrightarrow{\iota'} & \overline{Z} \\ \downarrow p_Z & & \downarrow p_Z & & \\ T_Z & \xrightarrow{\quad} & Z & \supseteq & \Delta \end{array}$$

Let  $\mathfrak{d} \subseteq \mathcal{O}(T_Z)$  be the vanishing ideal of  $\Delta \cap T_Z$ . For the vanishing ideal of  $\Delta$  in the Cox ring we obtain

$$I(\iota'(p_Z^{-1}(\Delta))) = I(\overline{\iota(p_Z^{-1}(\Delta \cap T_Z))}) = \sqrt{(\iota^*)^{-1}(p_Z^* \mathfrak{d})}.$$

We turn to i) and label the coordinates of

$$T_X = ((\mathbb{K}^*)^{n_1} \times (\mathbb{K}^*)^{n_1}) \times \dots \times ((\mathbb{K}^*)^{n_r} \times (\mathbb{K}^*)^{n_r})$$

by  ${}_rU_{ij}$  where  $r = 1, \dots, \mathbf{r}$ ,  $i = 1, \dots, n_r$  and  $j = n_r+1, n_r+2$ . Then the comorphism  $p_X^*$  of the corresponding Laurent polynomial rings is given as

$$p_X^*: \mathbb{K}[{}_rU_{ij}^{\pm}] \rightarrow \mathbb{K}[{}_rT_{ij}^{\pm}]; \quad {}_rU_{ij} \mapsto {}_rT_{ij} {}_rT_{0j}^{-1}.$$

The vanishing ideal of  $\Delta_X \cap T_X$  is generated by

$${}_rU_{i,n_r+1} - {}_rU_{i,n_r+2} \quad \text{where } r = 1, \dots, \mathbf{r}, \text{ and } i = 1, \dots, n_r.$$

Note that for any  $r = 1, \dots, \mathbf{r}$  and  $i, i' = 1, \dots, n_r$  this ideal also contains the elements

$${}_rU_{i,n_r+1} {}_rU_{i',n_r+1}^{-1} - {}_rU_{i,n_r+2} {}_rU_{i',n_r+2}^{-1}.$$

Pulling back all these equation via  $p_X^*$  yields the ideal  $\iota_X^*(\mathfrak{a}_X)$  in the Laurent polynomial ring  $\mathcal{O}(T_{\hat{X}})$ . Since  $\mathfrak{a}_X$  is an ideal of  $2 \times 2$ -minors, it is prime (in fact, it is the vanishing ideal of the Segre embedding). Hence Remark 2.6 gives our assertion.

We turn to ii) and proceed analogously. Here the coordinates of the dense torus  $T_Y = (\mathbb{K}^*)^n$  will be labeled  $U_i$  with  $i = 1, \dots, n$ . The comorphism  $p_Y^*$  is given by

$$p_Y^*: \mathbb{K}[U_i^\pm] \rightarrow \mathbb{K}[T_{ij}^\pm]; \quad U_i \mapsto T_{i,n+1} T_{i,n+2}^{-1}.$$

In  $\mathcal{O}(T_Y)$  the ideal of  $\Delta_Y \cap T_Y$  is generated by the relations  $U_i - U_j$  for  $1 \leq i < j \leq n$ . Pulling them back via  $p_Y^*$  yields the ideal  $\iota^* \mathfrak{a}_Y$  and the same argument as in i) yields the assertion.  $\square$

We denote the functions from Lemma 2.5 by  ${}_r f_{ij} \in \mathcal{O}(\overline{X})$  and  $g_{ij} \in \mathcal{O}(\overline{Y})$  where  $r$  corresponds to the  $r$ -th matrix and in both cases  $i, j$  define the respective columns. These functions  ${}_r f_{ij}$  and  $g_{ij}$  give rise to the stretched embeddings

$$\begin{aligned} \overline{\pi}_X: \bigoplus_{r=1}^{\mathbf{r}} \mathbb{K}^{2(n_r+1)} &\rightarrow \bigoplus_{r=1}^{\mathbf{r}} \left( \mathbb{K}^{2(n_r+1)} \oplus \mathbb{K}^{\binom{n_r+1}{2}} \right) \\ (x_1, \dots, x_{\mathbf{r}}) &\mapsto ((x_1, {}_1 f_{ij}(x_1)), \dots, (x_{\mathbf{r}}, {}_{\mathbf{r}} f_{ij}(x_{\mathbf{r}}))) \\ \overline{\pi}_Y: \mathbb{K}^{2n} &\rightarrow \mathbb{K}^{2n} \oplus \mathbb{K}^{\binom{n}{2}} \\ y &\mapsto (y, g_{ij}(y)). \end{aligned}$$

The vanishing ideals of the images are given by

$$\begin{aligned} I_{X,1} &:= \langle {}_r T_{ij} - {}_r f_{ij}; r = 1, \dots, \mathbf{r}, 0 \leq i < j \leq n_r + 2, i \leq n_r \rangle, \\ I_{Y,1} &:= \langle S_{ij} - g_{ij}; 1 \leq i < j \leq n + 2, i \leq n \rangle. \end{aligned}$$

We denote by  $\iota_{X,1}^*$ ,  $\iota_{X,2}^*$ ,  $\alpha_X^*$  and  $\iota_{Y,1}^*$ ,  $\iota_{Y,2}^*$ ,  $\alpha_Y^*$  the respective morphisms from Construction 2.3. The new Laurent polynomial rings are then given by

$$\begin{aligned} \mathbb{K}[T_\infty^\pm, {}_r T_{ij}^\pm; r = 1, \dots, \mathbf{r}, 0 \leq i < j \leq n_r + 2, i \leq n_r], \\ \mathbb{K}[S_{ij}; 1 \leq i < j \leq n + 2], \end{aligned}$$

where the additional variables are  $T_\infty$  and  $S_{n+1,n+2}$  respectively. We transfer the above ideals according to Construction 2.3, i.e. we set

$$\begin{aligned} I'_{X,2} &:= \langle \alpha_X^* (\iota_{X,1}^*(I_{X,1})) \rangle \\ &= \langle {}_r T_{ij} T_\infty - {}_r f_{ij}; r = 1, \dots, \mathbf{r}, 0 \leq i < j \leq n_r + 2, i \leq n_r \rangle, \\ I'_{Y,2} &:= \langle \alpha_Y^* (\iota_{Y,1}^*(I_{Y,1})) \rangle \\ &= \langle S_{ij} S_{n+1,n+2} - g_{ij}; 1 \leq i < j \leq n + 2, i \leq n \rangle. \end{aligned}$$

We first have to compute their preimages under  $\iota_{X,2}^*$  and  $\iota_{Y,2}^*$ , then we then show that  $T_\infty$  and  $S_{n+1,n+2}$  define prime elements and divide non of the remaining variables. Since the resulting relations are very closely related to the Plücker relations, we introduce some new notation. For this let  $0 \leq i, j, k, l \leq n$  be distinct integers. Then we denote by  $q(i, j, k, l)$  the corresponding Plücker relation; i.e. if  $i < j < k < l$  holds, then we set

$$q(i, j, k, l) := T_{ij} T_{kl} - T_{ik} T_{jl} + T_{il} T_{jk} \in \mathbb{K}[T_{ij}; 0 \leq i < j \leq n].$$

**Lemma 2.7.** *Let  $0 \leq i_0, j_0 \leq n$  be distinct integers. In the Laurent polynomial ring  $\mathbb{K}[T_{ij}^\pm; 0 \leq i < j \leq n]$  consider the ideal*

$$I := \langle q(i_0, j_0, k, l); 0 \leq k, l \leq n, i_0, j_0, k, l \text{ pairwise distinct} \rangle.$$

*Then for any pairwise distinct  $0 \leq i, j, k, l \leq n$  we have  $q(i, j, k, l) \in I$ .*

*Proof.* We first claim that for distinct  $0 \leq i, j, k, l, m \leq n$  we have

$$(*) \quad q(i, j, k, l), q(i, j, k, m), q(i, j, l, m) \in I \implies q(i, k, l, m) \in I.$$

For this we assume without loss of generality that  $i < j < k < l < m$  holds. The claim then follows from the relation

$$q(i, k, l, m) = \frac{T_{jk}}{T_{ij}} q(i, j, l, m) - \frac{T_{jl}}{T_{ij}} q(i, j, k, m) + \frac{T_{jm}}{T_{ij}} q(i, j, k, l) \in I.$$

Now consider distinct  $0 \leq \alpha, \beta, \gamma, \delta \leq n$ . If  $\{\alpha, \beta, \gamma, \delta\} \cap \{i_0, j_0\} \neq \emptyset$  holds, then  $q(\alpha, \beta, \gamma, \delta) \in I$  follows from the above claim (\*). So assume that  $\{\alpha, \beta, \gamma, \delta\}$  and  $\{i_0, j_0\}$  are disjoint. Applying (\*) to the three collections of indices

$$i_0, j_0, \alpha, \beta, \gamma; \quad i_0, j_0, \alpha, \beta, \delta; \quad i_0, j_0, \alpha, \gamma, \delta$$

shows that  $q(i_0, \alpha, \beta, \gamma)$ ,  $q(i_0, \alpha, \beta, \delta)$  and  $q(i_0, \alpha, \gamma, \delta)$  lie in  $I$ . Another application of (\*) then proves  $q(\alpha, \beta, \gamma, \delta) \in I$ .  $\square$

We are now ready to prove Theorem 1.2, for Theorem 1.1 we require some further preparations.

*Proof of Theorem 1.2.* Using Lemma 2.7 we see that the ideals  $\langle \iota_{Y,2}^* I_Y \rangle$  and  $I'_{Y,2}$  coincide. Since  $I_Y$  is prime from Remark 2.6 we infer that

$$(\iota_{Y,2}^*)^{-1} I'_{Y,2} = (\iota_{Y,2}^*)^{-1} \langle \iota_{Y,2}^* I_Y \rangle = I_Y.$$

Since  $I_Y$  is the ideal of Plücker relations,  $S_{n+1, n+2}$  is prime and does not divide any of the remaining variables. We determine the grading of the Cox ring. The ring  $\mathcal{O}(\bar{Y}) = \mathbb{K}[S_{ij}; i = 1, \dots, n, j = n+1, n+2]$  is  $\mathbb{Z}^n$ -graded by  $\deg(S_{ij}) = e_i$ . Under the stretched embedding the new variables  $S_{ij}$  where  $1 \leq i < j \leq n$  are assigned the degrees  $\deg(S_{ij}) = \deg(f_{ij}) = e_i + e_j$ . Finally, under the blow-up the weights are modified according to 2.3 to give the asserted grading.  $\square$

We turn to the remaining case of  $X = X' \times X'$ .

**Lemma 2.8.** *Let  $R := \mathbb{K}[T_\infty, T_1, \dots, T_n]$  be graded by  $\mathbb{Z}_{\geq 0}$  and let  $I \subseteq R$  be a homogeneous ideal. Suppose that  $T_\infty \notin \sqrt{I}$  and  $\deg(T_\infty) > 0$  hold. If the ideals  $I + \langle T_\infty \rangle$  and  $\sqrt{I}$  are prime, then so is  $I$ .*

*Proof.* Compare also [9, Proof of Theorem 1]. Since  $I + \langle T_\infty \rangle$  is a radical ideal, we have  $\sqrt{I} \subseteq I + \langle T_\infty \rangle$ . With this we obtain

$$\sqrt{I} = (I + \langle T_\infty \rangle) \cap \sqrt{I} = I + \langle T_\infty \rangle \sqrt{I}.$$

Note that for the second equality we used that  $\sqrt{I}$  is prime and  $T_\infty \notin \sqrt{I}$  holds. Let  $\pi: R \rightarrow R/I$  denote the canonical projection of  $\mathbb{Z}_{\geq 0}$ -graded algebras. Then we have  $\pi(\sqrt{I}) = \pi(\langle T_\infty \rangle \sqrt{I})$  and  $\deg(\pi(T_\infty)) > 0$ . The assertion follows from the graded version of Nakayama's Lemma.  $\square$

**Lemma 2.9** ([9, Proposition 4]). *Let  $1 \leq c \leq n$  be an integer. Then in the polynomial ring  $\mathbb{K}[T_{ij}; 0 \leq i < j \leq n+2]$  the following relations generate a prime ideal*

$$\begin{aligned} & -T_{ik}T_{jk} + T_{il}T_{jk}; \quad 0 \leq i < j \leq c < k < l \leq n+2, \\ & T_{ij}T_{kl} - T_{ik}T_{jk} + T_{il}T_{jk}; \quad 0 \leq i < j < k < l \leq n+2 \quad \text{different from above.} \end{aligned}$$

*Proof of Theorem 1.2.* First we claim that the ideal  $I_X$  is prime. For this note that the ideal  $\langle T_\infty \rangle + I_X$  is generated by  $T_\infty$  and the equations

$$\begin{aligned} & -{}_rT_{ik} {}_rT_{jk} + {}_rT_{il} {}_rT_{jk}; \quad 0 \leq i < j \leq n_r, \quad k = n_r + 1, \quad l = n_r + 2, \\ & {}_rT_{ij} {}_rT_{kl} - {}_rT_{ik} {}_rT_{jl} + {}_rT_{il} {}_rT_{jk}; \quad 0 \leq i < j < k < l \leq n + 2 \quad \text{diff. f. above} \end{aligned}$$

where  $r = 1, \dots, \mathbf{r}$ . From Lemma 2.9 we infer that  $\langle T_\infty \rangle + I_X$  is prime; we check the remaining assumptions of Lemma 2.8. Consider the classical grading of  $R_X$ , then  $I_X$  is homogeneous and  $\deg T_\infty > 0$  holds. We only have to verify that  $V(I_X)$  is irreducible. For this recall that we transferred the ideal  $I_{X,1}$  via

$$I'_{X,1} = \langle \iota_{X,1}^* I_{X,1} \rangle \quad \text{and} \quad I'_{X,2} = \langle \alpha_X^* I'_{X,1} \rangle.$$

Treating the index  $\infty$  as  $n_r + 1, n_r + 2$  in Lemma 2.7 we see that the latter ideal is given by  $I'_{X,2} = \langle \iota_{X,2}^* (I_X) \rangle$ . We track the respective zero sets.

$$V(I_X) = \overline{V(I'_{X,2})} = \overline{\alpha_X^{-1} V(I'_{X,1})} = \overline{\alpha_X^{-1} (\iota_{X,1}^{-1} (V(I_{X,1})) )}$$

Since  $\alpha_X$  has connected kernel and  $V(I_{X,1})$  is the graph of  $\overline{X}$  and as such irreducible, so is  $V(I_X)$ . This then implies that  $I_X$  is prime.

By Remark 2.6 this means that  $I_X = (\iota_{X,2}^*)^{-1} \langle \iota_{X,2}^* I_X \rangle = (\iota_{X,2}^*)^{-1} (I'_{X,2})$  holds. By Proposition 2.4 the only thing left to verify is that  $T_\infty$  does not divide any of the remaining variables. For this we compute the grading of the Cox ring; for reasons of degree it is then impossible for  $T_\infty$  to divide any other variable. The  $\mathbb{Z}^{\mathbf{r}} \times \mathbb{Z}^{\mathbf{r}}$ -grading of

$$\mathcal{O}(\overline{X}) = \mathbb{K}[{}_rT_{ij}; \quad r = 1, \dots, \mathbf{r}, \quad i = 0, \dots, n_r, \quad j = n_r + 1, n_r + 2]$$

is given by

$$\deg({}_rT_{ij}) = \begin{cases} (e_r, 0) & \text{if } j = n_r + 1, \\ (0, e_r) & \text{if } j = n_r + 2. \end{cases}$$

When stretching the embedding we add for every  $r = 1, \dots, \mathbf{r}$  the variables  ${}_rT_{ij}$  where  $0 \leq i < j \leq n_r$ . These are assigned the degrees  $\deg({}_rT_{ij}) = \deg({}_rg_{ij}) = (e_r, e_r)$ . Finally under the blow-up the degrees are modified according to Construction 2.3 to give the asserted grading.  $\square$

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